Exercises from MATH 2010D (2019/20)
(Problem set 4)

Ex 1. Determine whether each of the following limit exists, if yes, find its value; if no, prove your assertion.

(a)
$$\lim_{(x,y)\to(0,0)} \frac{x^3+x^2y+3xy^2+3y^3}{x^2+3y^2}$$

(b)
$$\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{2x^2+y^2}$$

(c)
$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^4}$$

(d)
$$\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1}$$ (Ex!)

Solution

1(a)

$$\lim_{(x,y)\to(0,0)} \frac{x^3+x^2y+3xy^2+3y^3}{x^2+3y^2}$$

For any (x,y), it can be written as

$(x,y) = (r\cos\theta, r\sin\theta)$ for some $r \geq 0$, $\theta \in [0, 2\pi]$

where $r = \sqrt{x^2+y^2}$

$$\frac{x^3+x^2y+3xy^2+3y^3}{x^2+3y^2}$$

$$= \frac{r^3\cos^3\theta + r^3\cos^2\theta\sin\theta + 3r^3\cos\theta\sin^2\theta + 3r^3\sin^3\theta}{r^2\cos^2\theta + 3r^2\sin^2\theta}$$

$$= r \cdot \frac{\cos^3\theta + \cos^2\theta\sin\theta + 3\cos\theta\sin^2\theta + 3\sin^3\theta}{\cos^2\theta + 3\sin^2\theta}$$

numerator : $|\cos^3\theta + \cos^2\theta\sin\theta + 3\cos\theta\sin^2\theta + 3\sin^3\theta|$

$\leq |\cos^3\theta| + |\cos^2\theta\sin\theta| + 3|\cos\theta\sin^2\theta| + 3|\sin^3\theta|$

$\leq 1 + 1 + 3 + 3 = 8$

denominator : $\cos^2\theta + 3\sin^2\theta$

$= 1 + 2\sin^2\theta \geq 1$

$$\left|\frac{\cos^3\theta + \cos^2\theta\sin\theta + 3\cos\theta\sin^2\theta + 3\sin^3\theta}{\cos^2\theta + 3\sin^2\theta}\right| \leq \frac{8}{1}$$

$$\therefore \left| \frac{x^3 + x^2 y + 3xy^2 + 3y^3}{x^2 + 3y^2} \right| \leq 8r = 8\sqrt{x^2+y^2}$$

(for any x, y)

$\rightarrow 0$ as $(x,y) \rightarrow 0$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^3 + x^2 y + 3xy^2 + 3y^3}{x^2 + 3y^2} = 0$$

1(b)

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + \sin^2 y}{2x^2 + y^2}$$

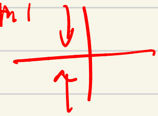
$$\boxed{= \frac{x^2 + \frac{\sin^2 y}{y^2} \cdot y^2}{2x^2 + y^2} \approx \frac{x^2 + y^2}{2x^2 + y^2}}$$

You can think in this way but don't write it out.

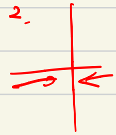
Does not exist :

Choosing two different paths approaching $(0,0)$ but with different limits,

path 1



path 2.



Path 1 : $x = 0$, $y \to 0$

Path 2 : $y = 0$, $x \to 0$

Path 1 : $\displaystyle \lim_{y\to 0} \frac{\sin^2 y}{y^2} = 1$

Path 2 : $\displaystyle \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$

$\therefore$ The limit does not exist.

1(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^4}$:

Put $x = r\cos\theta$, $y = r\sin\theta$

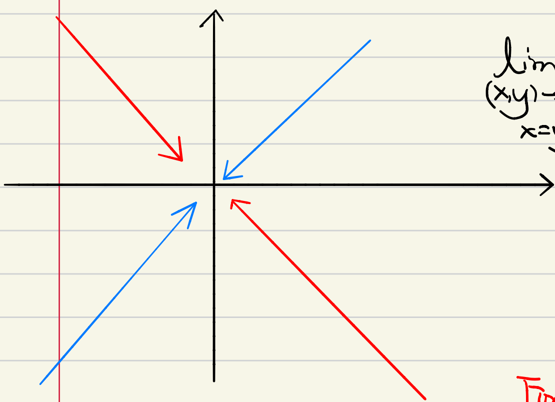
$$\frac{x^2y}{x^3+y^4} = \frac{r^3\cos^2\theta\sin\theta}{r^3\cos^3\theta + r^4\sin^4\theta}$$

$$= \frac{\cos^2\theta\sin\theta}{\cos^3\theta + r\sin^4\theta}$$

As $r\to 0$, $\displaystyle\lim_{r\to 0} \frac{\cos^2\theta\sin\theta}{\cos^3\theta + r\sin^4\theta} = \tan\theta$ (depends on θ)

Then, if you choose different paths approaching $(0,0)$, the limit will be different.

Example: For $\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$ (i.e $x=y$)



$$\lim_{\substack{(x,y)\to 0 \\ x=y}} \frac{x^2y}{x^3+y^4} = \lim_{x\to 0} \frac{x^3}{x^3+x^4}$$

$$= 1$$

$$\left(= \tan\frac{\pi}{4} \right)$$

For $\theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$ (i.e. $x=-y$)

$$\lim_{\substack{(x,y)\to(0,0) \\ x=-y}} \frac{x^2y}{x^3+y^4} = -1$$

Ex 2. Let $f(x,y) = \dfrac{xy^3}{x^3+y^5}$

(a)(i) Let $\gamma(t) = (t, mt)$, for $m \in \mathbb{R}$.
Show that $\lim_{t\to 0} f(\gamma(t)) = 0$.

(ii). Let $\gamma(t) = (0,t)$. Show that

$$\lim_{t\to 0} f(\gamma(t)) = 0$$

(b). Let $\gamma(t) = (t^3, t^2)$, for $m \in \mathbb{R}$. Show

that $\lim_{t\to 0} f(\gamma(t)) = 1$.

Hence, determine whether $\lim_{(x,y)\to(0,0)} f(x,y)$

exists or not.

(An example so that when you approach $(0,0)$ by any straight lines, you get the same limit. However the limit of the function does not exist.)

Solution

2(a)(i) $f(\gamma(t)) = \dfrac{m^3 t^4}{t^3(1+m^5t^2)} = \dfrac{m^3 t}{1+m^5t^2}$

$$\lim_{t\to 0} f(\gamma(t)) = 0$$

$\gamma(t) = (t, mt)$ is a parametrization of straight line with slope m.

(a)(ii) $\gamma(t) = (0, t)$ is the vertical line

$$\lim_{t\to 0} f(\gamma(t)) = 0$$

(b) Let $\gamma(t) = (t^3, t^2)$

$$f(\gamma(t)) = \frac{t^9}{t^9 + t^{10}} = \frac{1}{1+t}$$

$\therefore \lim_{t\to 0} f(\gamma(t)) = 1$

$\therefore$ Together with 2(a)(i), $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

Does not exist.

Explanation: From 2(a)(i), it shows that around $(0,0)$, there are points (x_0, y_0) so that $f(x_0,y_0)$ is close to 0

From 2(b), it shows that around $(0,0)$, there are points (x_1, y_1) so that $f(x_1,y_1)$ is close to 1

So, you cannot say that $f(x,y)$ is close to some specific number (i.e. the limit), when (x,y) is near 0.

Compare to Ex 1 (a)

For Ex 2, if you put $x = r\cos\theta$, $y = r\sin\theta$,

then
$$f(x,y) = \frac{r^4 \cos\theta \sin^3\theta}{r^3(\cos^3\theta + r^2\sin^5\theta)}$$

$$= \frac{r\cos\theta\sin^3\theta}{\cos^3\theta + r^2\sin^5\theta}$$

For any small $r > 0$, we may choose θ depending on r, so that $\cos\theta$ is even smaller than r.

Then, $f(x,y)$ may be away from 0 even when (x,y) is close to $(0,0)$ [i.e. r small]

e.g. $\theta = \frac{\pi}{2} - r$

$$\cos\theta = \cos\left(\frac{\pi}{2} - r\right) = \sin r \approx r$$
$$\sin\theta = \sqrt{1 - \cos^2\theta} \approx \sqrt{1 - r^2}$$

$$f(x,y) \approx \frac{r^2\left(\sqrt{1-r^2}\right)^3}{r^3 + r^2\left(\sqrt{1-r^2}\right)^5}$$

$$= \frac{\left(\sqrt{1-r^2}\right)^3}{r + \left(\sqrt{1-r^2}\right)^5}$$

$$\longrightarrow 1 \quad \text{as} \quad r \to 0$$

(Here we choose θ depending on r !)